TD 12 : Path and cycle 3-coloring, spanning tree, graph size

1 Path 3-coloring

We first recall the definition of an oriented path.

Definition (Oriented path)**.** In an oriented path, each process has at most one predecessor and at most one successor. Messages can be sent by processes in both directions.

Problem 1.1. In the $O(\log^* n)$ algorithm (which we shall refer to as the Cole-Vishkin algorithm) done during the class, it was shown how the number of colors reduce from m to $2\lfloor \log_2 m \rfloor + 1$ for $m \geq 5$, thus eventually reducing the number of colors to 6 (assuming that the final colors are $0, \ldots, 5$). Show how to further reduce this number to 3. Do the processes need to know the number of nodes in the graph beforehand ?

† **Problem 1.2.** The Cole-Vishkin algorithm done during the class works with oriented paths. Give an algorithm that is almost as fast and works even on non-oriented paths. You can assume that everyone has unique identifiers.

† **Problem 1.3** (Randomized and fast)**.** The simple randomized 3-coloring algorithm done during the class finds a 3-coloring in time $\mathcal{O}(\log n)$ with high probability, and it does *not* need any unique identifiers. Give a fast randomized algorithm that does this in $o(\log n)$ time. You can assume that the processes know the number of nodes n.

†† **Problem 1.4** (Oblivious algorithm)**.** Show that the Cole-Vishkin algorithm can be adapted to work even if the processes do not know the number of nodes in the path beforehand. You may assume that the path is oriented.

2 Cycle 3-coloring

Assume that there are n processes that form an oriented cycle C_n (as for paths, every process has at most one predecessor and at most one successor).

Problem 2.1. Show that the randomized and Cole-Vishkin algorithms used for 3-coloring paths can also be used for 3-coloring cycles.

† **Problem 2.2.** You may assume that the Cole-Vishkin algorithm actually has $\log^* n + \mathcal{O}(1)$ round complexity. Show that there is in fact an algorithm with $\frac{1}{2}\log^* n + \mathcal{O}(1)$ round complexity.

††† **Problem 2.3** (Nonexistence of constant-time algorithm)**.** For all $t \geq 0$, every t-round algorithm fails to 3-color some cycle. *Hint :* One proof of this relies on Ramsey's theorem on hypergraphs given below. Determine an appropriate choice of $k, c, n_1, \ldots, n_c$ in order to prove the lower bound.

Theorem (Ramsey)**.** *For any integers k and c and any integers $n_1, \ldots, n_c$, there is an integer $R = R(n_1, \ldots, n_c; k)$ such that if all size-k subsets of a set X where $|X| = R$ are colored with c different colors, then there exists $i \in [c]$ and a subset $X' \subseteq X$ where $|X'| = n_i$ such that all size-k subsets of X' are colored i.*

3 Graph size and Minimum Spanning Tree

† **Problem 3.1.** Give an algorithm that allows the processes to calculate a spanning tree (i) when there is a designated leader whose ID all processes know, and (ii) when there is no designated leader.

Problem 3.2. Give an algorithm for the processes to calculate the total number of vertices in the graph (i) when there is a designated leader whose ID all processes know, and (ii) when there is no designated leader.